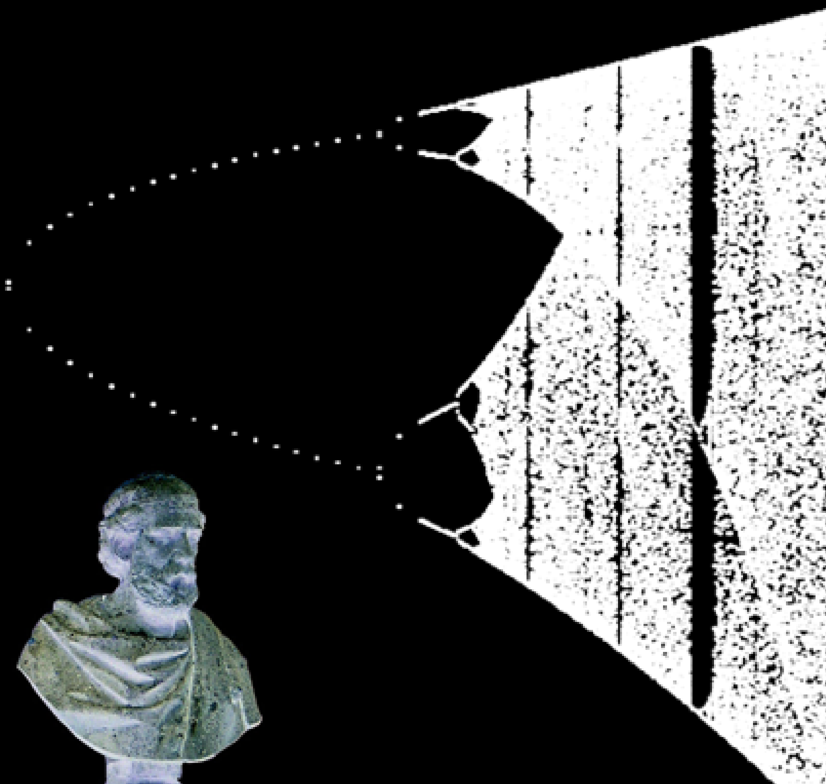
Archimedes confronts The Logistic Map
and the Basins of Attraction in …

The Collection Controversy



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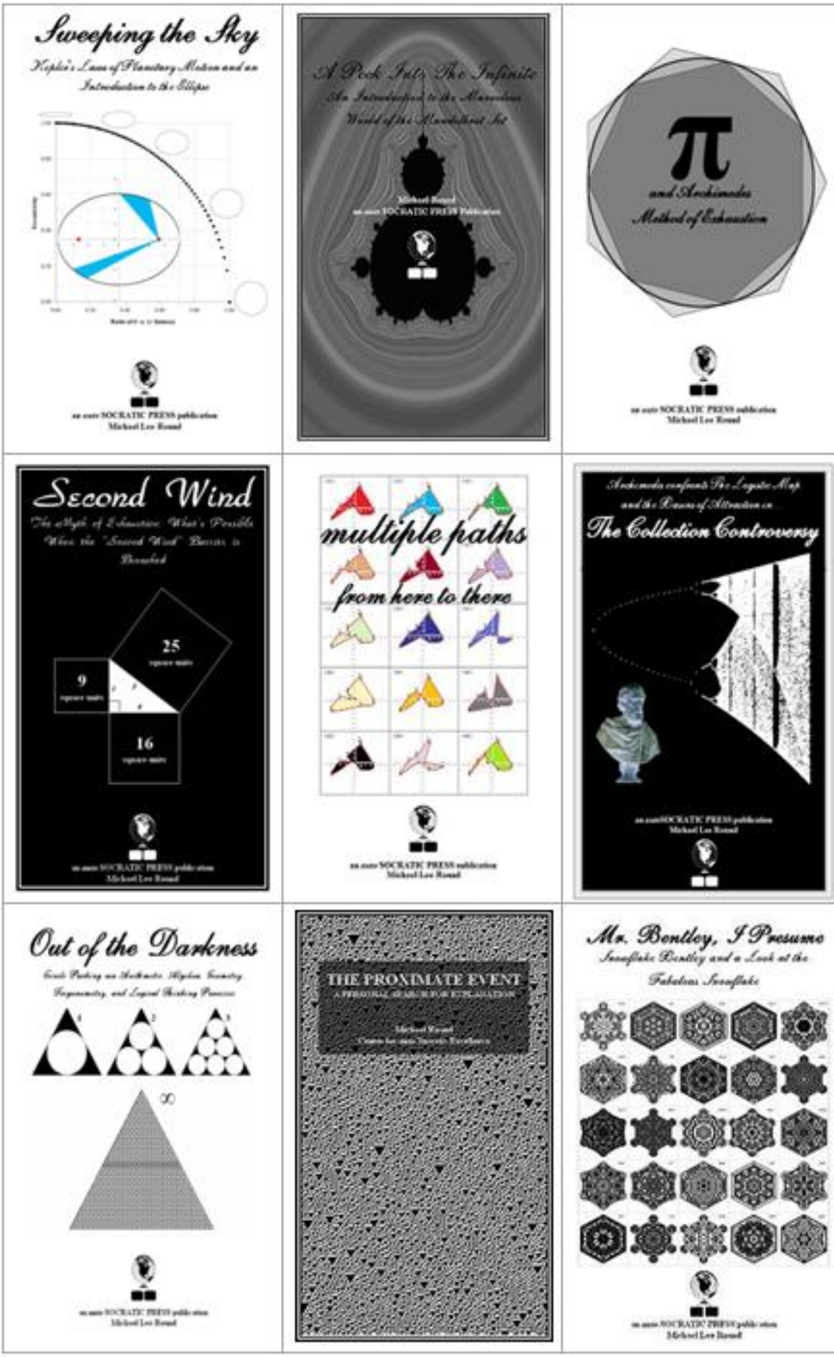
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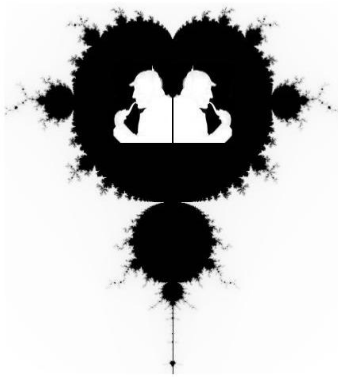

=EQUALS=

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PART I

SIMILAR PROBLEMS MEAN SIMILAR SOLUTIONS, RIGHT?

The scene: lunch at a small deli in Syracuse, Sicily. Mike Mason is eating lunch. The great Archimedes approaches, striding across the floor.

Archimedes:

Mr. Mason! How good to see you! I see you're dining alone. May I join you?

Mason: (Shoving some papers to the side of the table)

It would be my pleasure.

(Archimedes takes notice of the strange diagrams on the papers brushed aside).

Archimedes:

It's been a while since our last encounter. My ideas on buoyancy have been greatly enhanced since your last cross-examination! What are you working on there, my friend?

Mason:

One person's "cross-examination" is another person's "logical analysis"!

Archimedes:

Touché!

Mason:

I'm doing some work predicting the population of Canadian showshoe rabbits and lynx, and frankly, having a bit of trouble with it.

Archimedes:

Snowshoe rabbits and lynx? You *do* get around, from King's Crowns to predator-prey models! What's the issue?

Mason:

This table is the issue:

Year	Rabbit	Year	Lynx
1845	12,000	1848	28,000
1850	76,000	1853	8,000
1855	68,000	1858	24,000
1860	8,000	1863	4,000
1865	152,000	1868	60,000
1870	16,000	1873	8,000
1875	84,000	1878	32,000
1880	12,000	1883	16,000
1885	120,000	1888	72,000
1890	60,000	1893	20,000
1895	20,000	1898	36,000
1900	8,000	1903	4,000
1905	64,000	1908	40,000
1910	28,000	1913	4,000
1915	12,000	1918	24,000
1920	8,000	1923	2,000
1925	72,000	1928	28,000
1930	12,000	1933	8,000
1935	80,000	1938	32,000

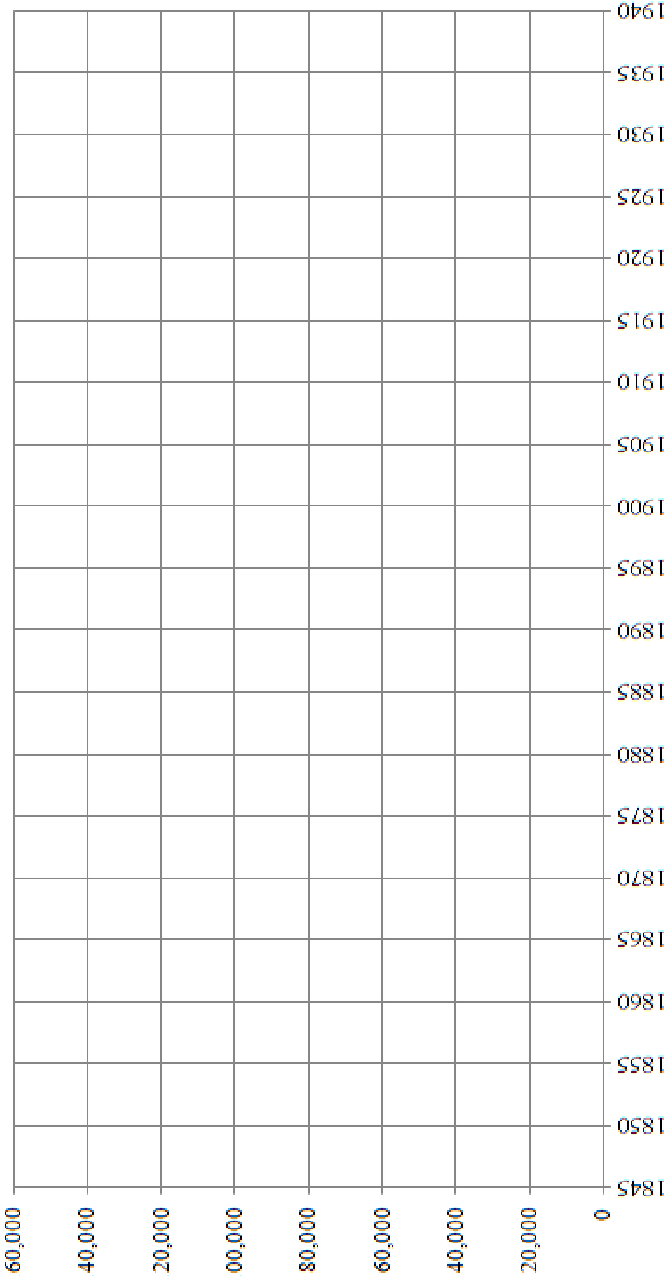
Archimedes:

What is this?

Mason:

These are the snowshoe rabbit and lynx populations in Canada over nearly a century.

Graph these and you'll see what I'm talking about.


1845
1850
1855
1860
1865
1870
1875
1880
1885
1890
1895
1900
1905
1910
1915
1920
1925
1930
1935
1940
0
20,000
40,000
60,000
80,000
100,000
120,000
140,000
160,000

Archimedes:

It looks pretty regular to me.

Mason:

It *is* pretty regular!

Archimedes:

Then what's the problem?

Mason:

I can't figure out *why* is it regular?

Archimedes:

And what have you found?

Mason:

Some interesting things. For example, let's suppose there are a lot of snowshoe rabbits. If there are a lot of rabbits, then the lynx will eat well. Right?

Archimedes:

Right.

Mason:

And if the lynx eat well, they will be able to survive and give birth to more lynx.

Archimedes:

It follows.

Mason:

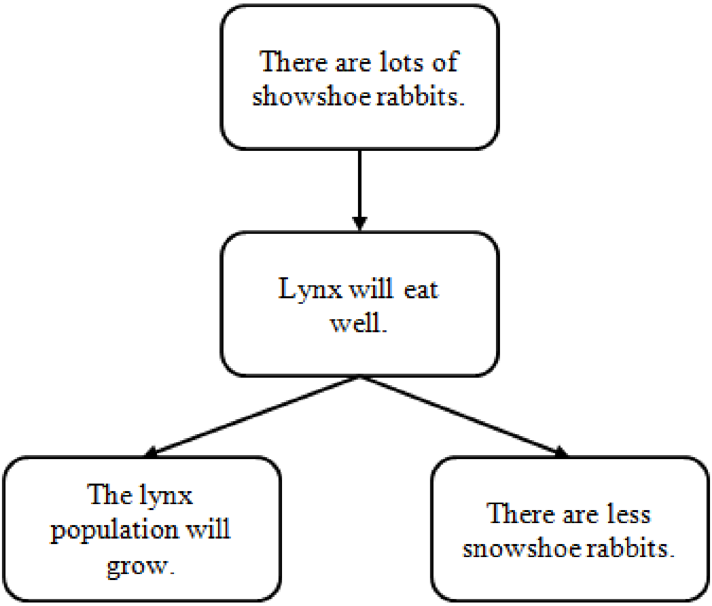
However, if the lynx are eating more and more rabbits, what happens to the rabbit population?

Archimedes:

It should go down.

Mason:

Exactly. So far, we've described this:



Archimedes:

I'm with you, though I don't see any problems.

Mason:

Let me continue. If the lynx population grows, and if there are less snowshoe rabbits, then what happens to the lynx population?

Archimedes:

It will fall.

Mason:

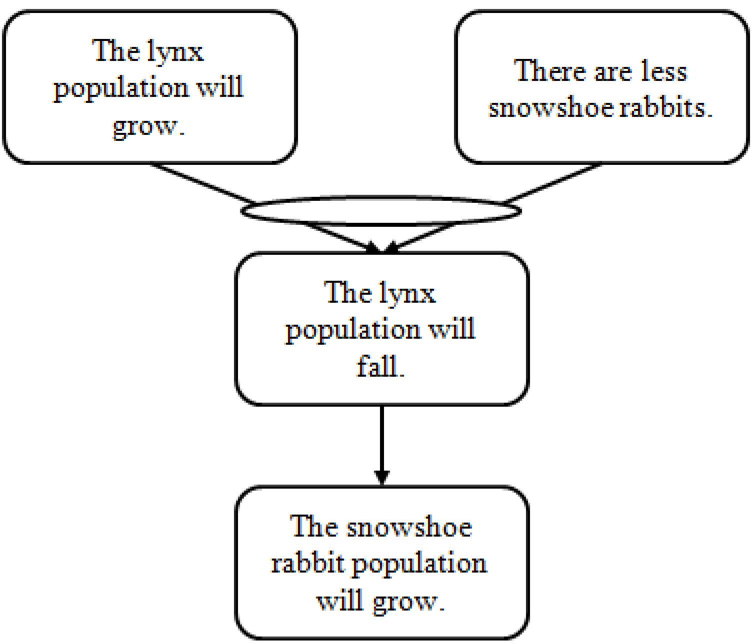
And if the number of lynx falls, what will happen to the snowshoe rabbit population?

Archimedes:

It has a chance to replenish itself. It will grow!

Mason:

Exactly. So far, we've described this:

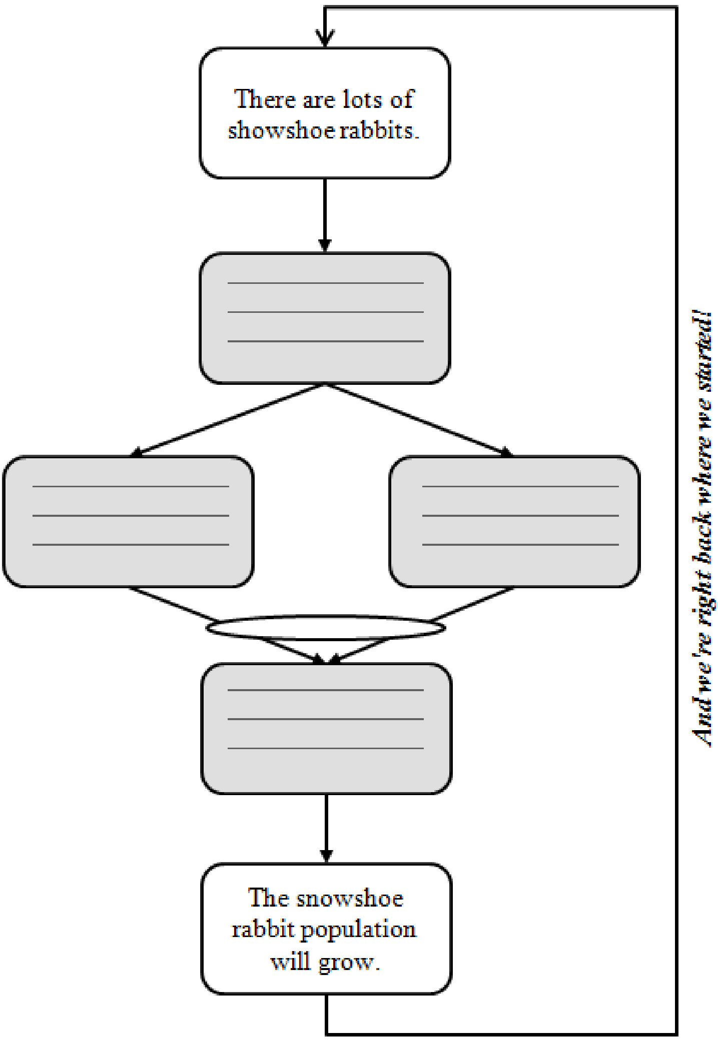


Archimedes:

I *still* don't see any problems!

Mason:

Let me finish! And if the snowshoe population grows, what happens? *We're right back where we started!*



Archimedes:

Is that a bad thing?

Mason:

I don't know if it's good or bad, but it's interesting! I guess my problem is I can't figure out *why* it oscillates like this.

I tried to make this even simpler by thinking only about snowshoe rabbits. Forget the lynx. Just rabbits. What would this population look like over time?

Archimedes:

And?

Mason:

For example, if *no* baby rabbits live, the rabbit population will eventually die. However, if *all* baby rabbits live, eventually there are too *many* rabbits, the environment cannot sustain them, and likely the population *again* does not survive.

Archimedes:

So there's some point in-between that surely does the job.

Mason:

That's what I thought as well, but there's some strange things I can't explain. Let …

Archimedes:

Nonsense. It's somewhere between the two ends. In fact, this sounds similar to something I'm working on.

Mason: (seeing it's no use talking about his own issue anymore)

What is it, my friend, you're working on?

Archimedes:

I've been working on a problem involving the circumference of a circle.

Mason (with a look of boredom):

And this is *exciting*? Are you going to tell me there's a "Eureka" moment wrapped up in *circles*?

Archimedes:

After the firm warning from the Judge in my last outburst, I've found it's best just to yell "*I have found it!*" to myself.

Mason:

Let's get to it, then.

Archimedes:

The other day, I put a pole in the middle of my yard, attaching one end of a string to the pole, and holding taught the other end. I then walked about the pole, making a circle.

Mason:

Sounds plenty boring thus far!

Archimedes:

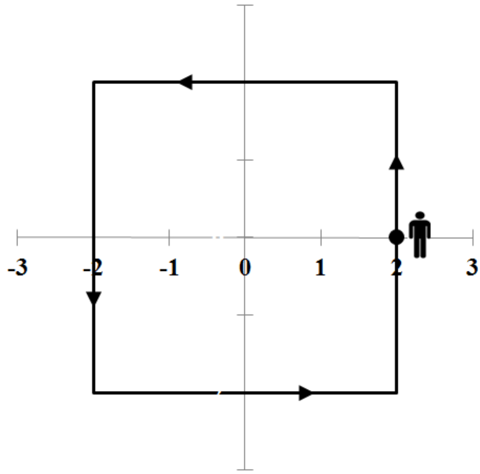
I thought so, too, until I asked myself: "*How far have I walked?*"

Mason:

I don’t understand.

Archimedes:

You see, had I walked in the shape of a *square*, I could figure out exactly how far I walked. For example, in this square, the length of each side is four.



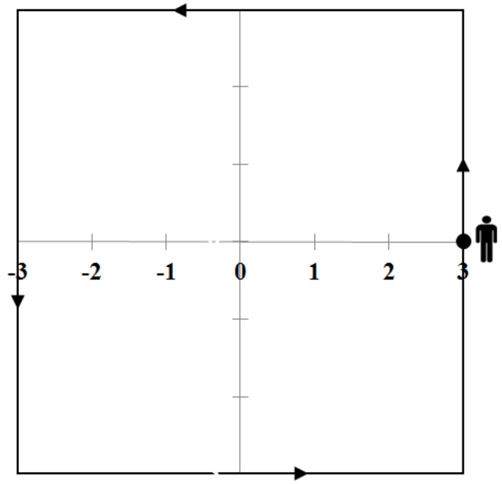
The total distance walked, then, = ________.

Mason:

Fair enough.

Archimedes:

And this square, with each side length 6, has a perimeter = ________.

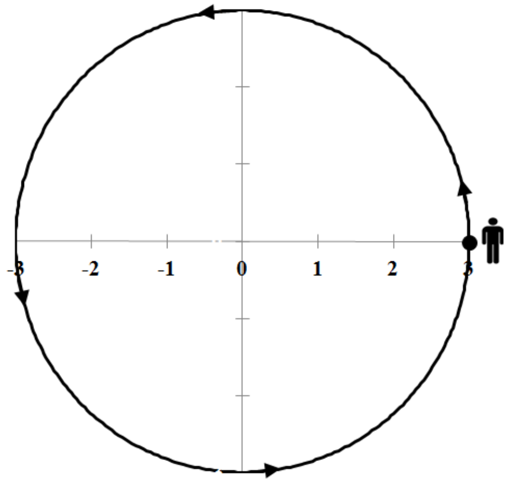


Mason:

OK.

Archimedes:

So how far have I walked in *this* situation?



Mason:

Off the top of my head I don't know, but there must be some easy answer.

Archimedes:

I thought so, too, but I can't find one! This has given me more trouble than determining the authenticity of the King's Crown!

Mason:

Who, by the way, is your client in *this* case?

Archimedes:

I'm my own client, as usual. Just me, trying to find answers to what should be simple questions.

Mason:

I thought only a fool employs himself as his attorney, but by the look in your eyes, I trust you've found the answer?

Archimedes:

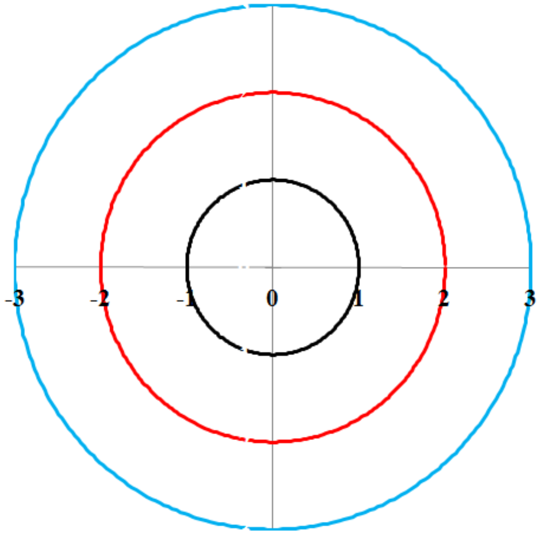
Not exactly, but I have found a way to *narrow* the answer.

Mason:

I don't understand.

Archimedes:

I couldn't figure out the relationship between the length of my rope and the distance I walked, and this nagged me. I drove my neighbors crazy walking around the yard, with different lengths of rope, tracing paths in the dirt. Here's the results for three circles:



THE THREE CIRCLES

	Small	Medium	Large
radius:	1.0	2.0	3.0
diameter:	2.0	4.0	6.0
my circumference:	6.0	13.0	19.0
ratio of circumference to diameter:			

As you see, the shorter the rope (radius), the shorter the circumference. The longer the rope (radius), the longer the circumference.

Mason:

That's obvious, isn't it?

Archimedes:

Of course it's obvious, but the question is, "How *much* longer?" What's the relationship between the length of the rope and the circumference of the circle?

The ratio of the circumference to the diameter goes from 3.0 to 3.3 to 3.2. And these were just three circles I walked. I actually walked many more, and as I did, my estimation of this ratio becomes more exact, and I start to close in on the ratio: 3.14.

Mason:

So that's the answer. What's the problem?

Archimedes:

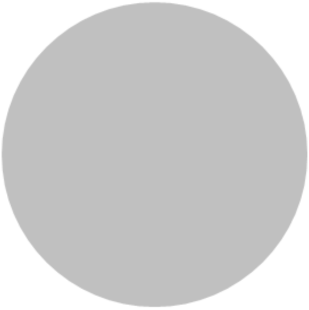
As I said, I've only found a way to *narrow* the answer. The ratio isn't *exactly* 3.14, only *about* 3.14.

Mason:

What do you mean "*narrow* the answer"?

Archimedes:

Look at this example: suppose I want to find the circumference of this circle:



Mason:

Fine.

Archimedes:

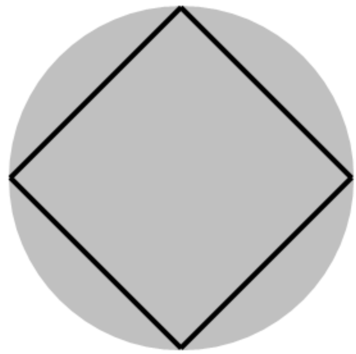
I don't know how to do that. However, remember I *do* know how to find – *exactly* – the perimeter of a straight-lined polygon.

Mason:

So?

Archimedes:

So I can embed such a figure inside the circle, right?

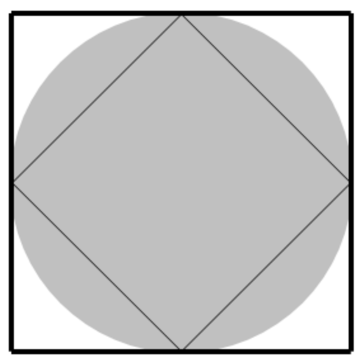


Mason:

Yes, but that doesn't seem to give you a very good approximation.

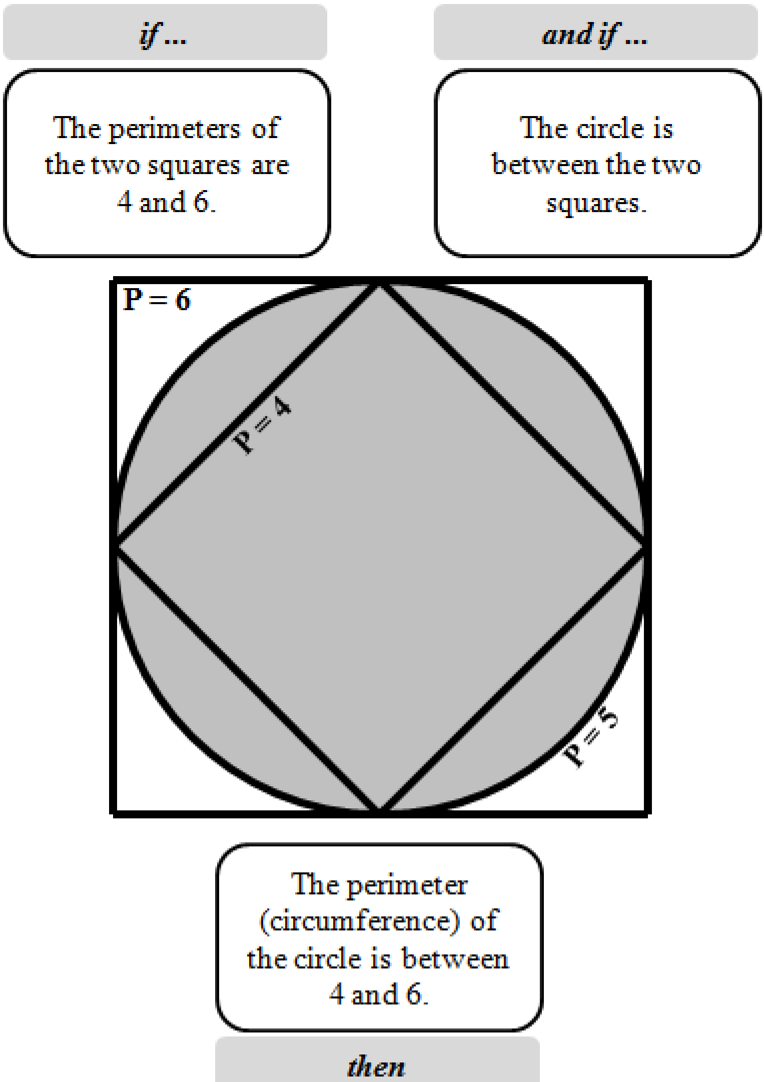
Archimedes (annoyed):

Of course not. The circumference is larger than the perimeter of the inscribed square. But I can also put the circle *inside* a larger square …



Mason (interrupting):

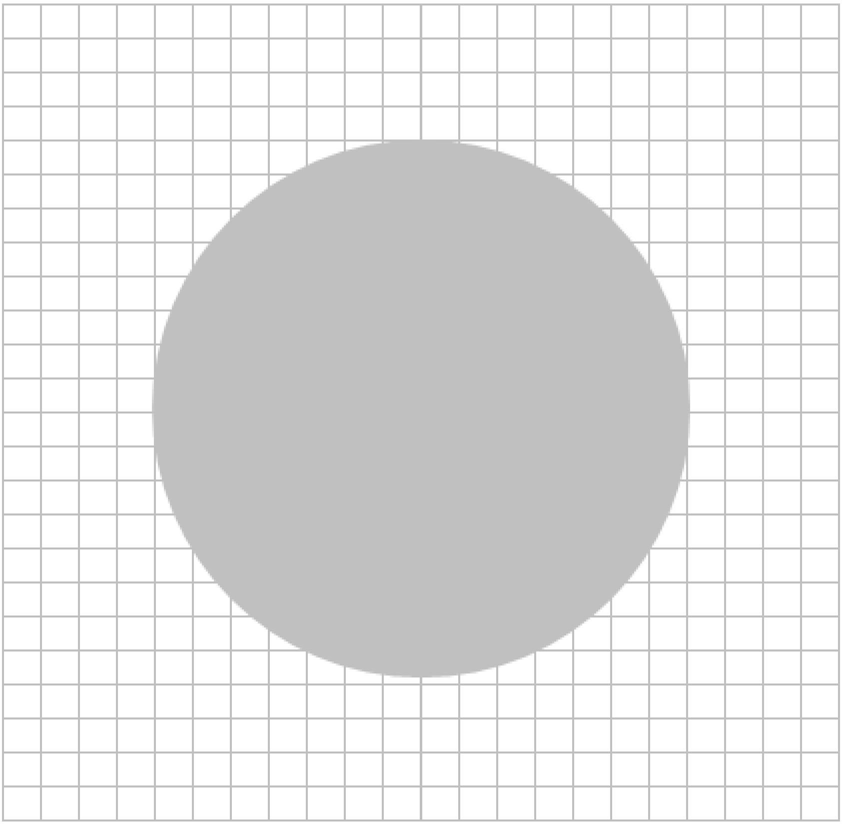
I see it! You're sandwiching the answer! The circumference of the circle is *between* the two perimeters. Do you just take an average of the two perimeters to approximate the circumference of the circle?



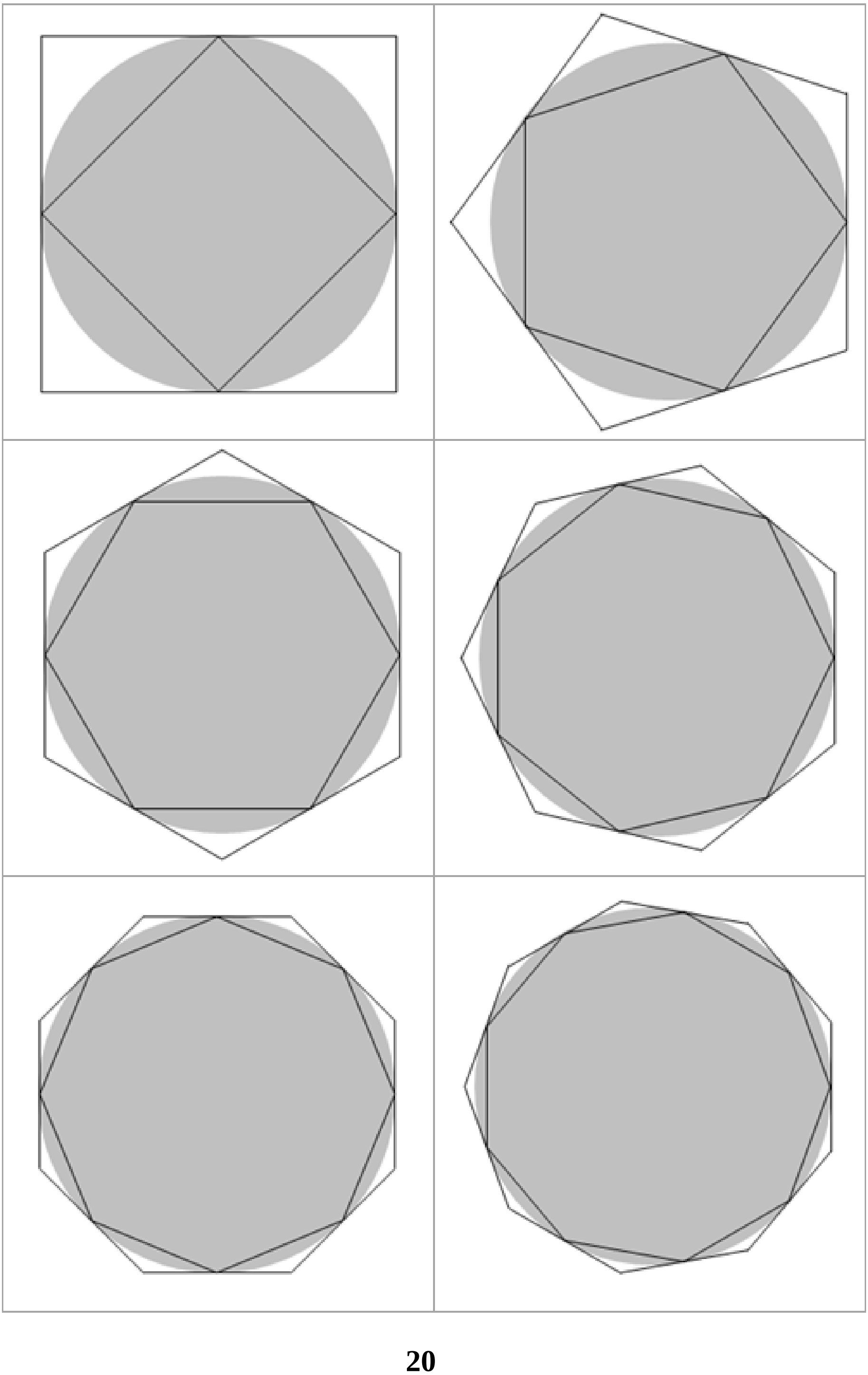
Archimedes (again, annoyed):

That's true – if I stopped at polygons with four sides. *But why do I have to stop at 4?* What happens to our problem "gap" if I increase the number of sides of the polygon?

Try it yourself, first embedding a regular pentagon in the circle, and then a regular pentagon outside the circle (like we did with squares):



Archimedes then methodically drew polygons with sides 4, 5, 6, 7, 8, and 9 sides).

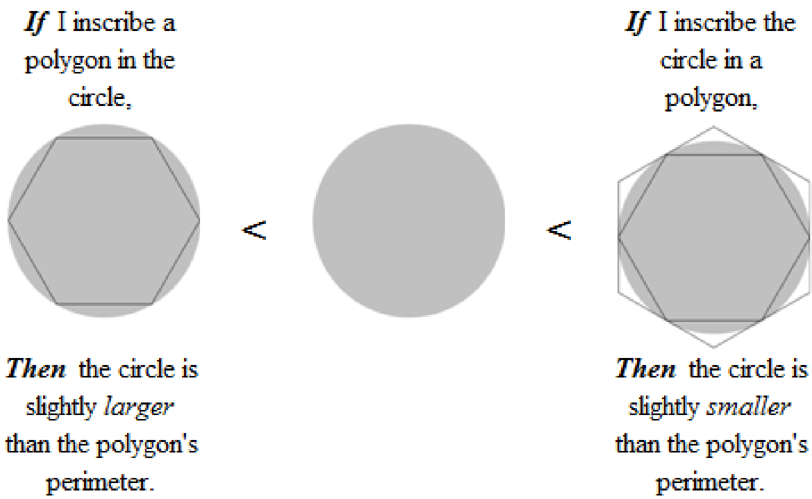


Mason:

The gap erodes! Very interesting. What do you call this method?

Archimedes:

You've already said part of it. I'm "sandwiching" the estimation between two perimeters I *do* know. That is:



Mason:

What's the *other* part?

Archimedes:

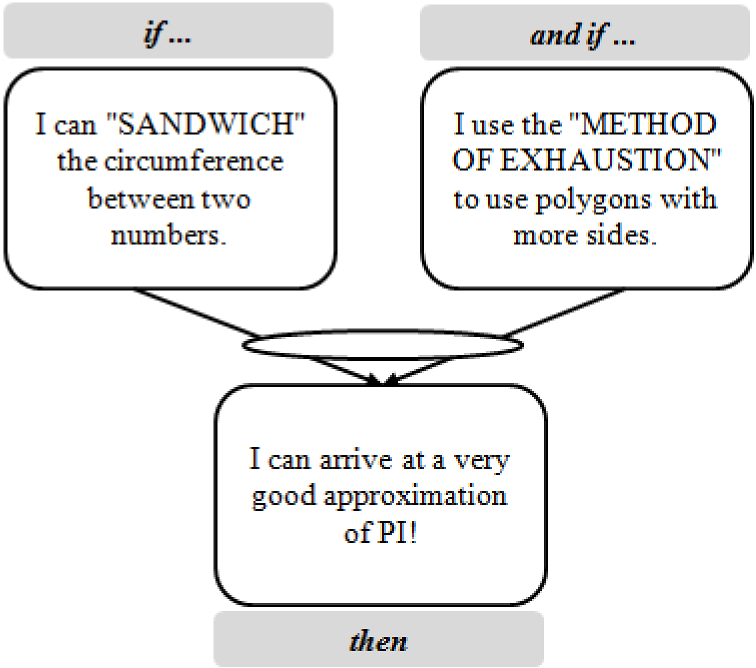
To get a better approximation, it takes more and more sides of the figures. Therefore, I call this the "Method of Exhaustion".

Mason:

I see what you mean, but there's something missing here …

Archimedes:

Nonsense. These two things combined give us a very good approximation of π. That is:



Now, let’s stop talking about “Sandwiching the Solution” and order our *real* sandwiches! I’ve worked up an appetite!

Mason (to himself):

(Maybe if I wait until we’re eating – I’ll get a chance to voice my concerns).

(end of Part I)

PART II

HOW CAN THIS BE?

Mason (with mouth half-full, while both eat):

Let's get back to my problem. I was working on the idea of modeling the life – and death – of lynx and snowshoe rabbits. Because I was having trouble with that, I tried to simplify the model, working just with rabbits – and food.

Archimedes:

And?

Mason:

The logic of an ecology with just *one* animal is remarkably similar to that of two!

Archimedes:

I don't believe it.

Mason:

I didn't either, at first. Suppose there are too many rabbits in the environment. What happens to the food supply?

Archimedes:

It goes down.

Mason:

And as the food supply goes down – that is, there is less food – what happens to the population of rabbits?

Archimedes:

It goes down.

Mason:

And as the rabbit population decreases, what happens to the available food supply?

Archimedes:

It has a chance to replenish itself!

Mason:

And as it replenishes itself, what happens to the rabbit population?

Archimedes:

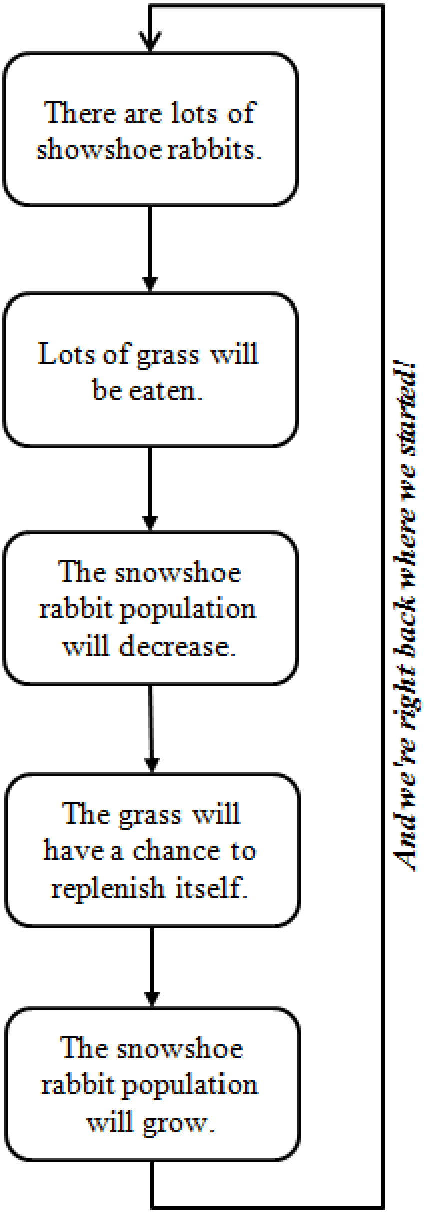
It grows! I see the circular logic now!

Mason:

Do you?

Archimedes:

Yes! And this explains the ups and downs of the earlier graph!



Mason:

Let’s suppose it does. Now let’s try and model it.

Archimedes:

OK. Go ahead.

Mason:

That's where I run into trouble. I don't know how.

Archimedes:

What do you mean, "You don't know how?" *We just did it!* What's the problem?

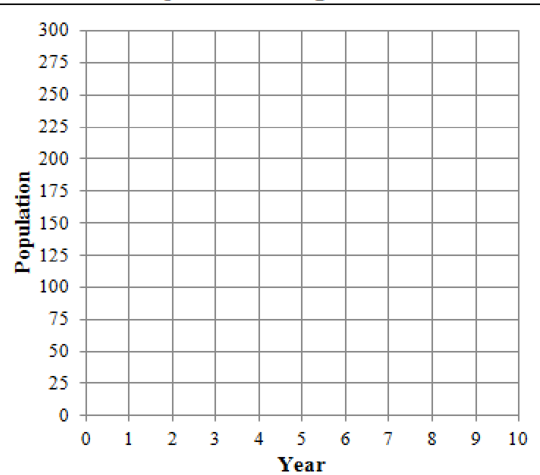
Mason:

I'll show you. Let's suppose I start with 100 rabbits. I immediately run into trouble. Whether I say they increase from year to year, decrease, or stay the same, I can't get the population to alternate like our earlier graph. Take a look yourself:

ANNUAL CHANGE
Population Starting at 1.00

Year	10% ↓	10% ↑	No Change
0	100	100	100
1	90	110	100
2	80	120	100
3	70	130	100
4	60	140	100
5	50	150	100
6	40	160	100
7	30	170	100
8	20	180	100
9	10	190	100
10	0	200	100

LINEAR CHANGE
Population Starting at 100



Archimedes:

I see what you mean. But this is with “linear” change, where the amount of change is the same from year to year. What if we change this to “exponential” change, where the change compounds?

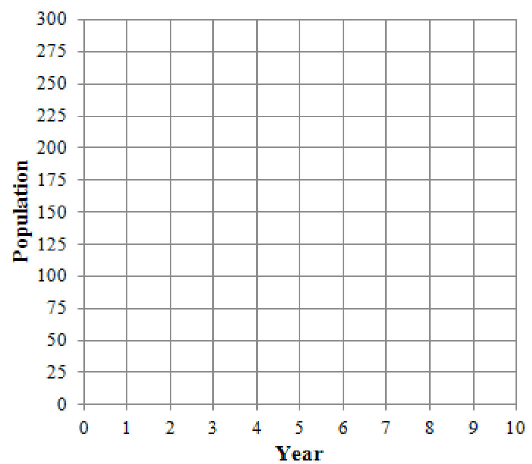
Mason:

Like this?

ANNUAL CHANGE
Population Starting at 1.00

Year	10% ↓	10% ↑	No Change
0	100	100	100
1	90	110	100
2	81	121	100
3	73	133	100
4	66	146	100
5	59	161	100
6	53	177	100
7	48	195	100
8	43	214	100
9	39	236	100
10	35	259	100

EXPONENTIAL CHANGE
Population Starting at 100



Archimedes:

OK – it makes a difference, but it’s still the case once you *get* moving in a direction, you *keep* moving in that direction.

Mason:

The problem seems to be: if the population one year is down, then the next year it’s likely the population will go up (because of more grass). However, our formula doesn’t show this.

Archimedes:

That’s obvious – we already said that. What we’re trying to do now is show it mathematically.

Mason:

How about this. If the probability of death is low, that means there’s likely a lot of grass around. So the fact the probability is low doesn’t contain all the information.

Suppose, however, we call one thing “life” and label it with the variable p. Then “death” is equal to $1 - p$. If we combine them, we get $p(1 - p)$.

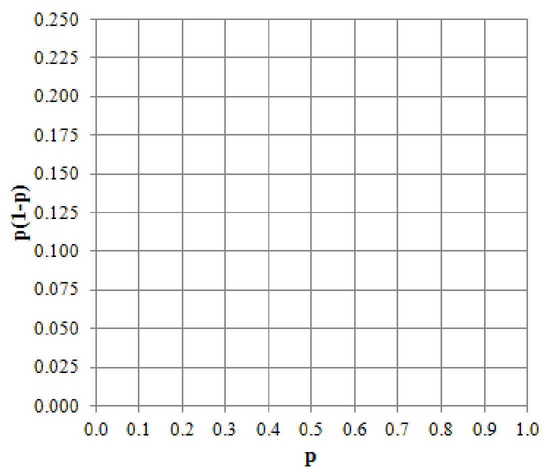
Archimedes:

What does that get us?

Mason:

Let’s take a look:

p	(1-p)	p(1-p)
0.00	1.00	0.00
0.10	0.90	0.09
0.20	0.80	0.16
0.30	0.70	0.21
0.40	0.60	0.24
0.50	0.50	0.25
0.60	0.40	0.24
0.70	0.30	0.21
0.80	0.20	0.16
0.90	0.10	0.09
1.00	0.00	0.00

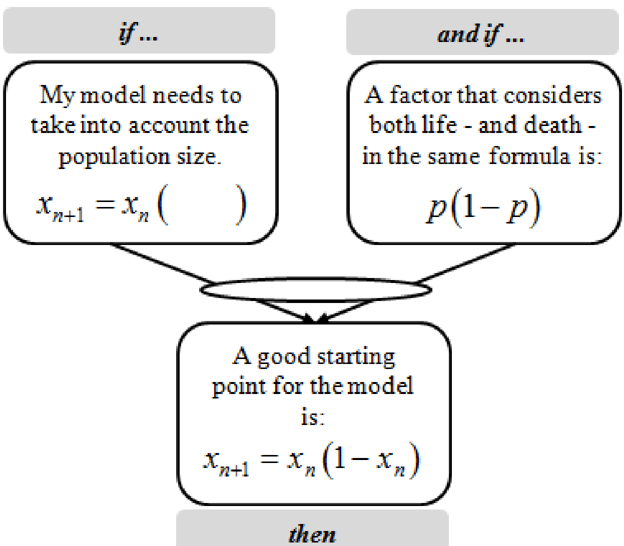


Archimedes:

What does this mean for our model?

Mason:

It does one crucial thing. Remember, our model needed, firstly, to take into account the population size. We've done that. We also needed some factor that took into account life – and death – in the same formula. This would allow us – hopefully – to see the results go up and down like we saw in the first graph. We've accomplished this:



Archimedes:

This looks good, but we must be missing something.

Mason:

What do you mean?

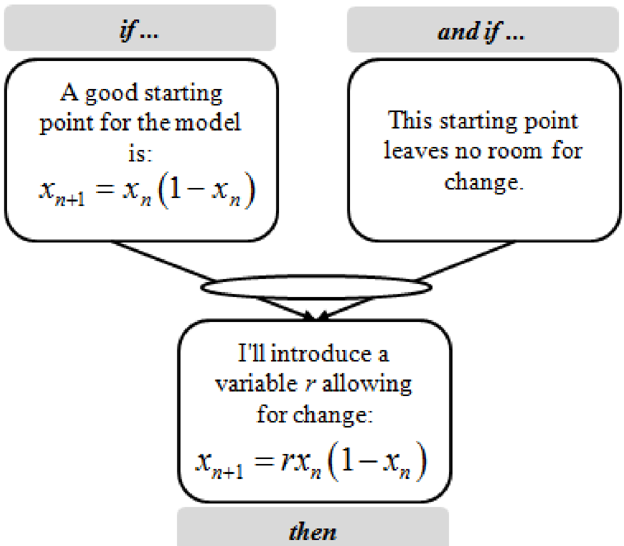
Archimedes:

There's no variety here. We get the same numbers – all the time!

Mason:

You're right. Suppose we add another variable, call it *r*, that serves the purpose our initial 10% increase/decrease did. *Will that work?*

That is:



Archimedes:

So our modeling formula is:

$$x_{n+1} = rx_n(1 - x_n)$$

Mason:

That's it!

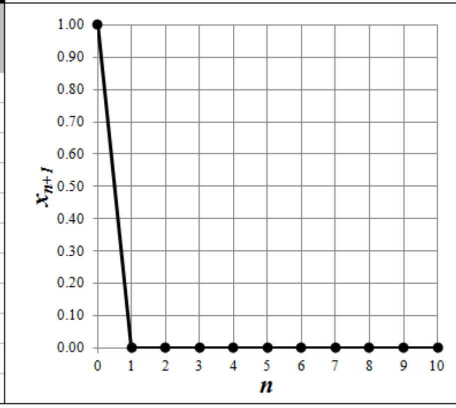
Archimedes:

Sounds good. But you know my need for practicality. Let's put some numbers on the table. Can you show me some examples?

Mason:

I agree. Let's start with a simple one: Let's set the initial population value equal to 1.0 (like we did above), and let the parameter *r* = 1.00.

r =	1.00
x_0 =	1.00
n	$x_{n+1} = rx_n(1-x_n)$
0	1.000
1	0.000
2	0.000
3	0.000
4	0.000
5	0.000
6	0.000
7	0.000
8	0.000
9	0.000
10	0.000



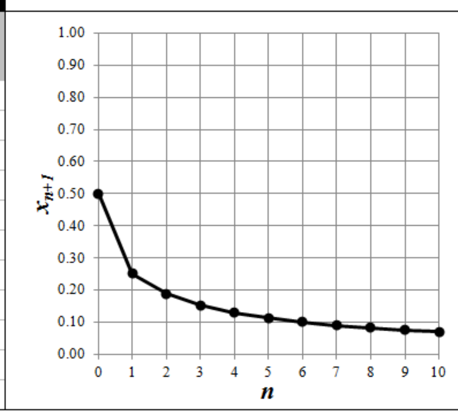
Archimedes:

Not a very promising future for this species!

Mason:

I see the problem: with the starting population of x_0=1.00, the other part of the formula (1 – *p*) becomes zero. Once it's zero, it can never escape. Let's try a starting population of x_0=0.50, then.

r =	1.00
x_0 =	0.50
n	$x_{n+1} = rx_n(1-x_n)$
0	0.500
1	0.250
2	0.188
3	0.152
4	0.129
5	0.112
6	0.100
7	0.090
8	0.082
9	0.075
10	0.069



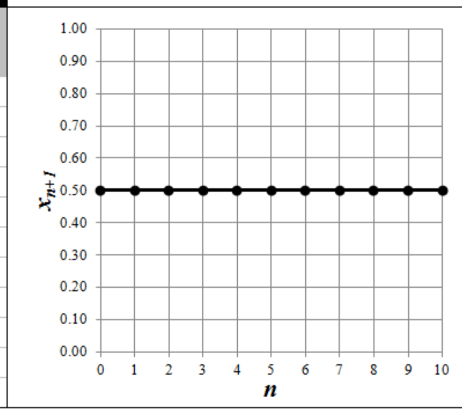
Archimedes:

This looks a little better, but I only see the population going down, like we saw earlier with our exponential growth graph.

Mason:

Let's see what happens with r = 2.00:

r =	2.00
x_0 =	0.50
n	$x_{n+1} = rx_n(1-x_n)$
0	0.500
1	0.500
2	0.500
3	0.500
4	0.500
5	0.500
6	0.500
7	0.500
8	0.500
9	0.500
10	0.500



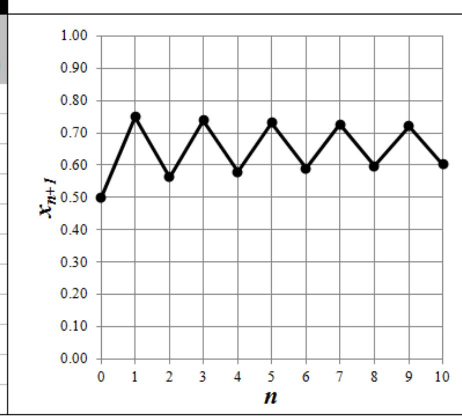
Archimedes:

It's changing, but not looking good. Where's the ups and downs we were hoping for? Try r = 3.00:

Mason:

Here you go:

r =	3.00
x_0 =	0.50
n	$x_{n+1} = rx_n(1-x_n)$
0	0.500
1	0.750
2	0.563
3	0.738
4	0.580
5	0.731
6	0.590
7	0.726
8	0.597
9	0.722
10	0.603

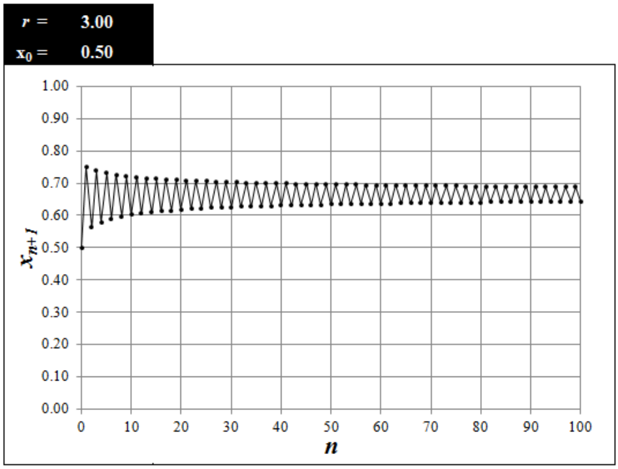


Archimedes:

Now we're talking! And this is for only 10 years. What happens if we extend this out to 100 years?

Mason:

Let's see:



Archimedes:

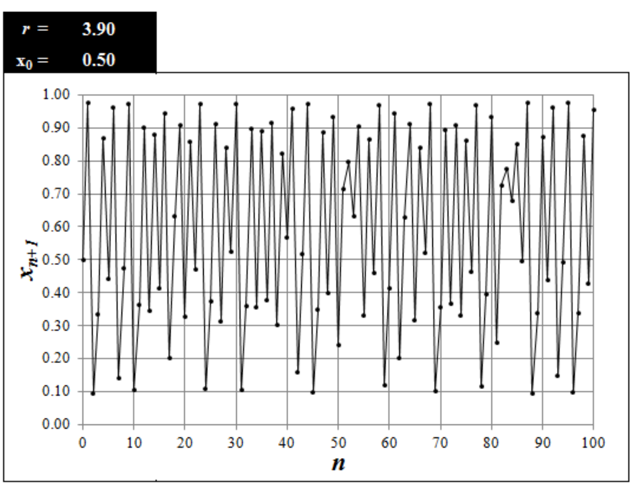
This is our model, then. It works! The population does not die out, and there's some variation in the population from year to year! *r* = 3.00 must be it!

Mason (disappointed):

Aren't you a bit curious what happens when *r* increases even more?

Archimedes:

You've got me there. What happens when we increase the value of *r*?

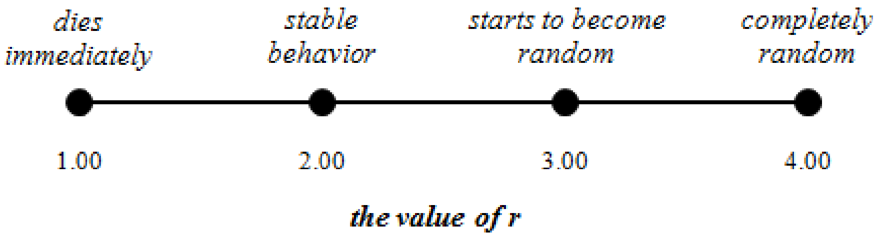


Mason:

Here's the behavior when *r* = 3.90. Look at all the variation!

Archimedes:

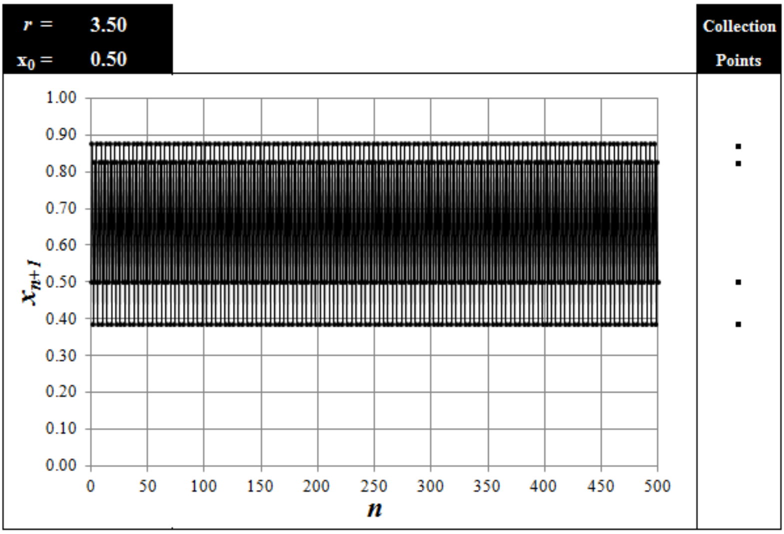
I discern the pattern immediately.



I need to run an errand – I'll be back in a few minutes.

Mason:

Sounds good – that will give me a chance to play around with this a bit. *To himself:* one thing that's easy to see is the variability. One thing that's hard to see is where the process starts to repeat itself. Maybe I can create a separate graph of "Collection Points". For example, with r = 3.50, there are four collection points:



Mason: (again, to himself)

I've been playing around with different values of *r*, and collection points, and have found something odd. I see Archimedes returning. Let's see what he has to say about it.

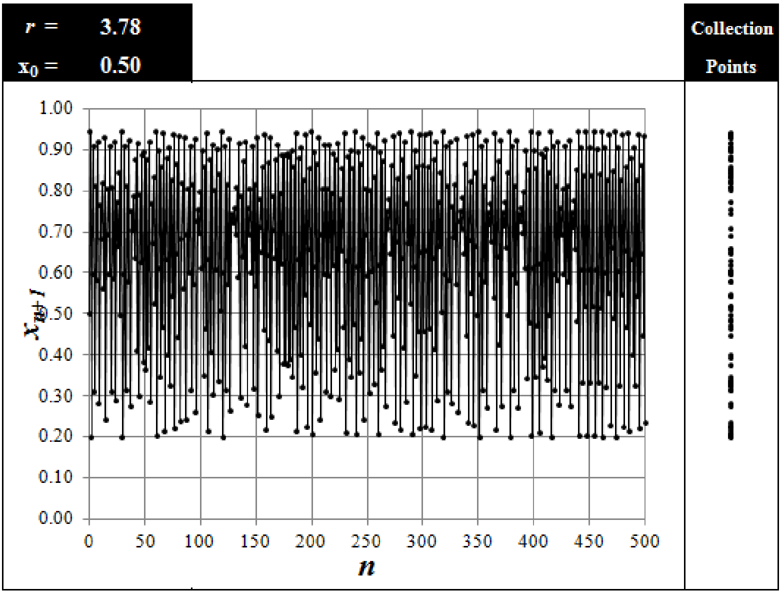
Archimedes: (seeing the new graph)

I see you've added something to our project. What is this all about?

(Mason tells Archimedes about the idea of a "Collection Point" graph).

Mason:

I've got a question for you. Let's suppose I use *r* = 3.78. The graph is as follows:



Archimedes:

OK – pretty messy, but exactly as I predicted. As *r* increases, the randomness increases.

Mason:

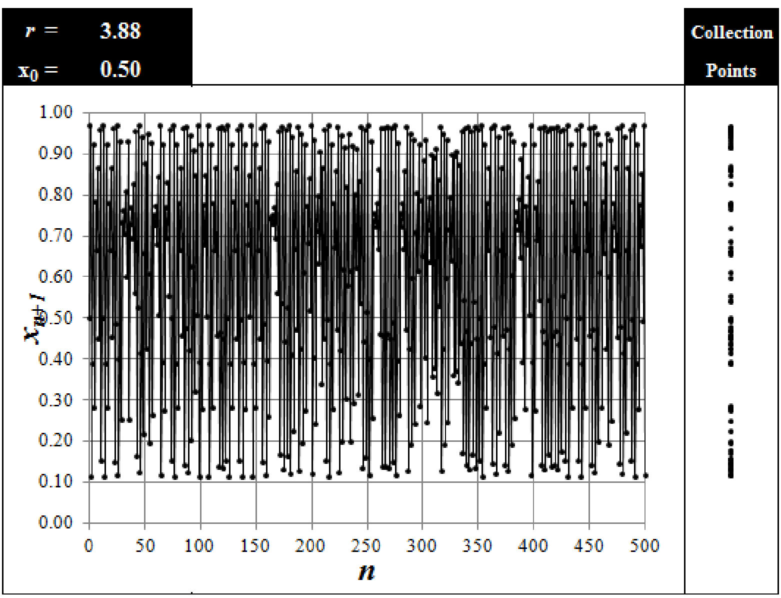
And if I increase the value of *r* to *r* = 3.88, what do you think will happen?

Archimedes:

Even more randomness.

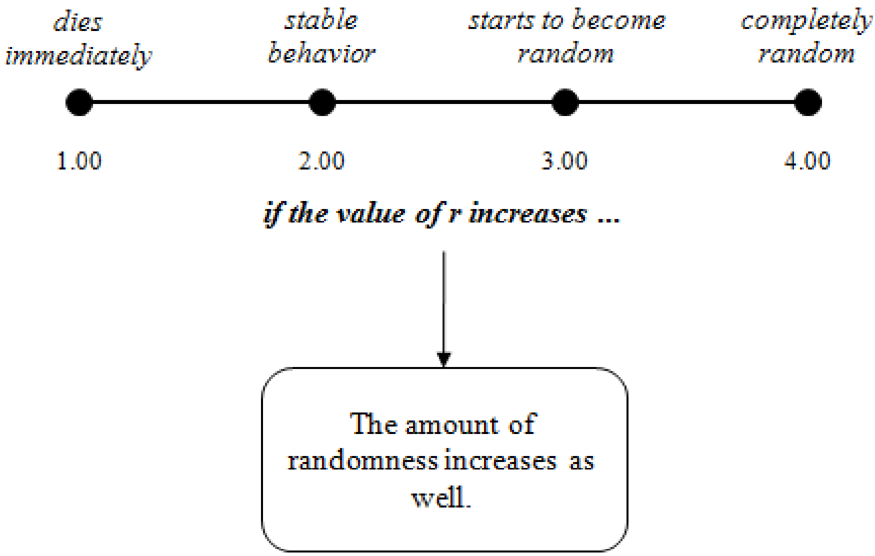
Mason:

Let's see.



Archimedes:

The great Archimedes, right again! Let me iterate my theory:



Mason:

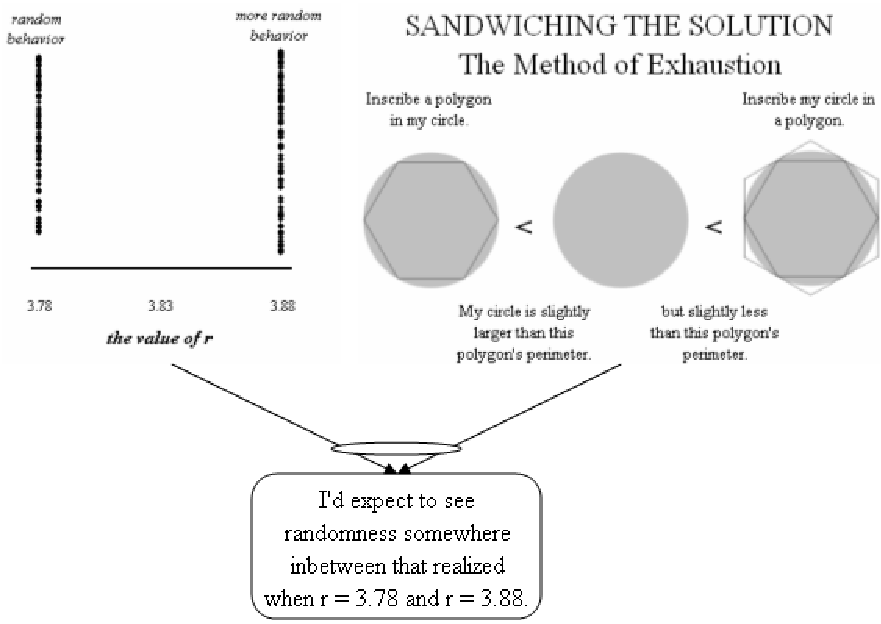
Indeed! It is exactly as you theorized. But I want to make clear your theory. If we've got *some* randomness at 3.78 and *more* randomness at 3.88, and if your "Sandwich Theorem" from the Method of Exhaustion is correct, then we'd expect to see "in-between" randomness at 3.83, right?

Archimedes:

Right.

Mason:

To put it visually, we have:

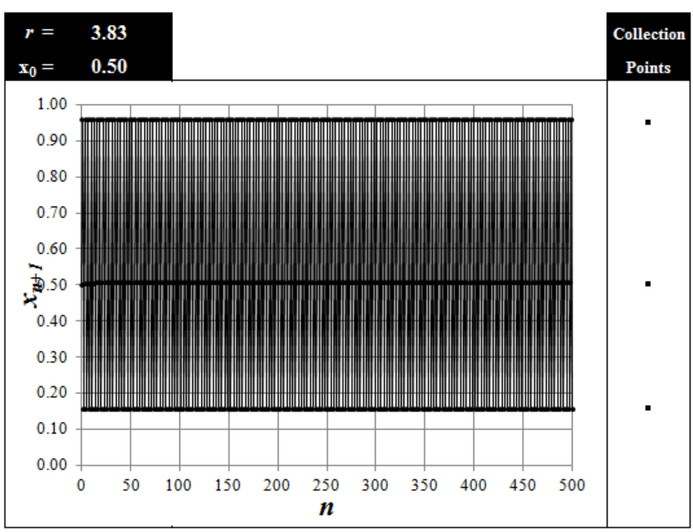


Archimedes:

Agreed.

Mason:

Here's what I've got:

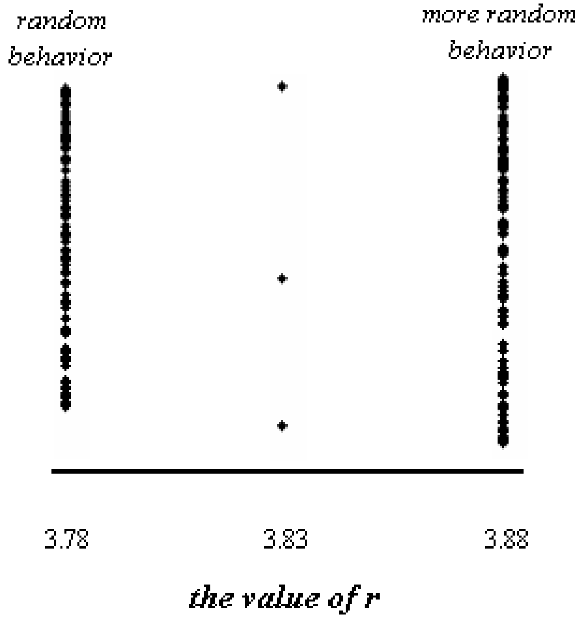


Archimedes:

Wait a minute – that doesn't make any sense …

Mason: (not stopping):

And if I plot the results for 3.78, 3.83, and 3.88 side by side, I get this:

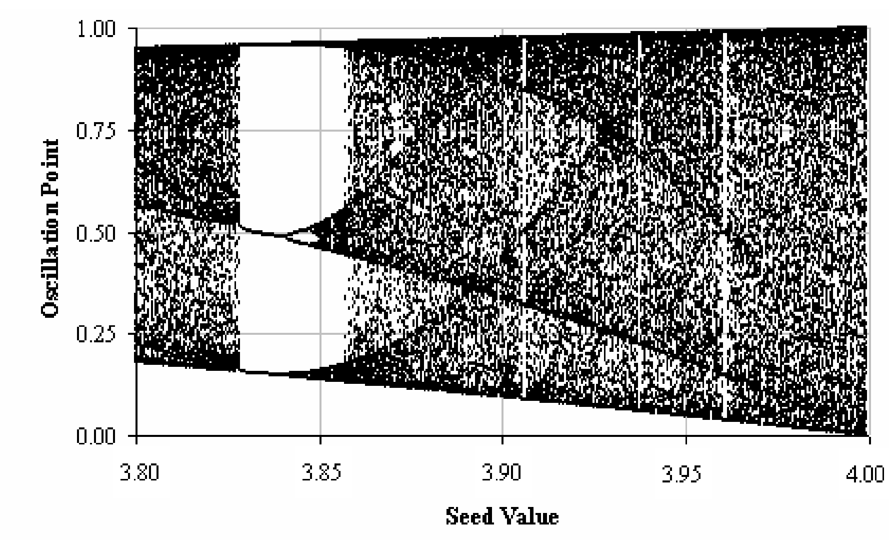


Archimedes:

How can this be? There must be …

Mason (on a roll):

And If I plot not just these values, but run the program for *hundreds* of values of *r* between [3.78, 4.00], I get *this!*



Archimedes:

Would you stop! Let me breathe! What is the meaning of this?

Mason:

I have no idea, but you wouldn't listen to me!

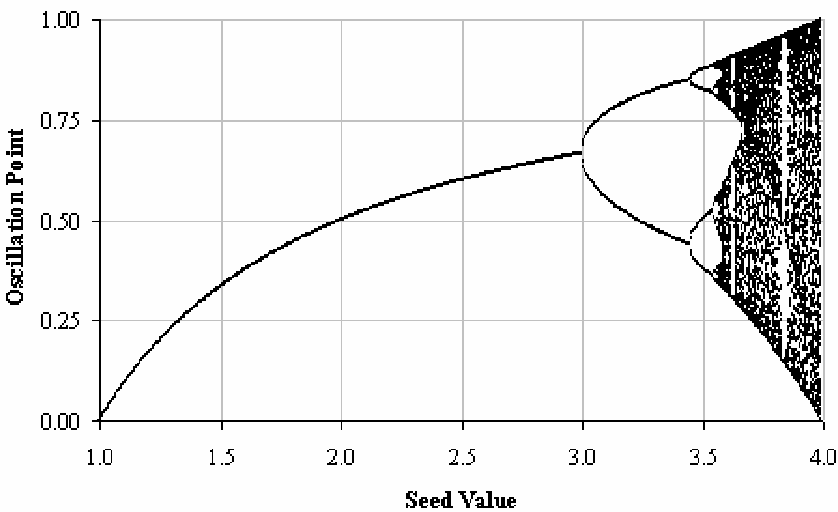
Archimedes:

I suddenly know how Pythagoras must have felt when he was asked the length of the diagonal of a square with sides of length 1.

Now you've got me curious: what happens when you plot this for *all* values of *r*, and not just these between [3.80, 4.00]?

Mason:

I'm glad you asked! It looks like this:



Archimedes:

This is an unusual map. And your counter-example seems to have disproved the validity of my Theorem. So what are we saying here? Are you saying my "Sandwich Theorem" is wrong?

Mason:

Maybe. Maybe not. Maybe something's missing. It seemed to work for your circumference problem. However, it didn't work at all in the population growth example. *Something's missing somewhere!*

Archimedes:

Let me take a stab at verbalizing it: I had a pretty good grasp on how polygons behave. There's no surprises – that I know of. In circumstances like this, the Sandwich Theorem works.

Mason:

But how can we be sure what *"pretty good grasp*" means? After all, we thought we had a *"pretty good grasp"* of how our simple equation worked, *and we were wrong!*

Archimedes:

Maybe it's enough just to have the knowledge we may not know what we think we know!

Mason:

Maybe it's the case "Common Sense is Sometimes Not Common Practice".

Archimedes:

Maybe. And I'll pick up the tab for the sandwiches!

Mason:

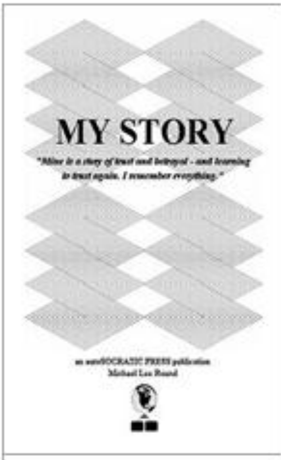
No. I'll get the tab, if you will show me how you created these graphics!

Archimedes:

Deal!

PARTS III & IV

COMING


MY STORY
"Mine is a story of trust and betrayal - and learning to trust again. I remember everything."
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From Dot to Dot
and everything in-between.
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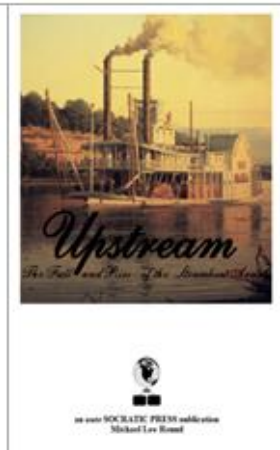
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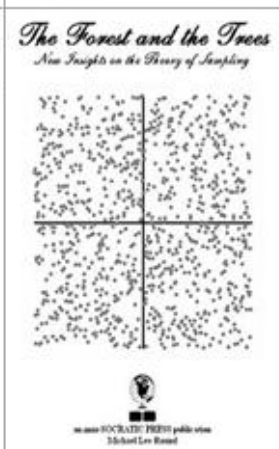
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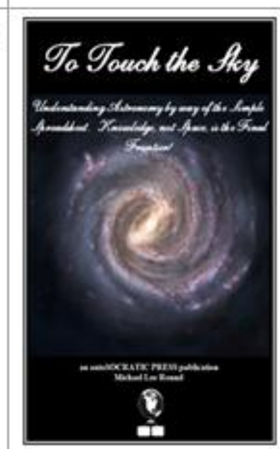
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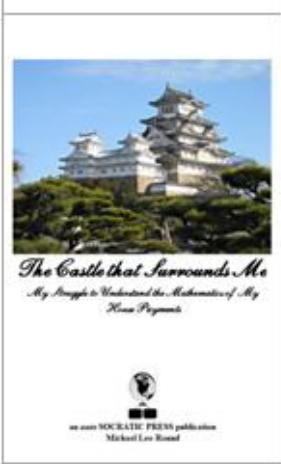
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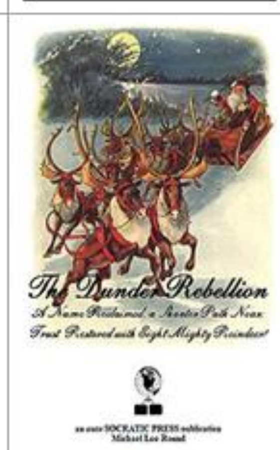
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Mobius

Mobius

Mobius

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