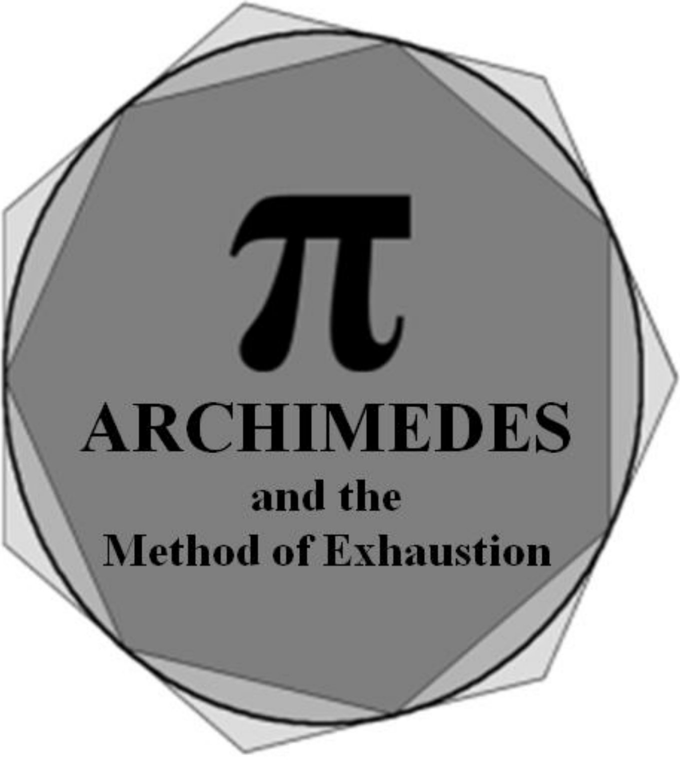
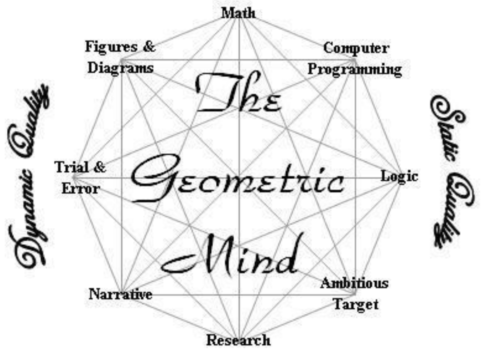
THE GEOMETRIC MIND SERIES
an *auto*SOCRATIC QUICK-START publication



π ARCHIMEDES
and the
Method of Exhaustion



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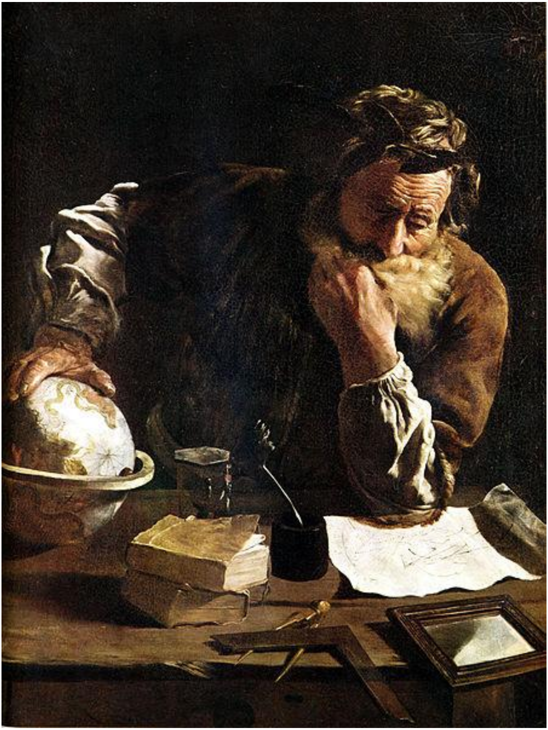
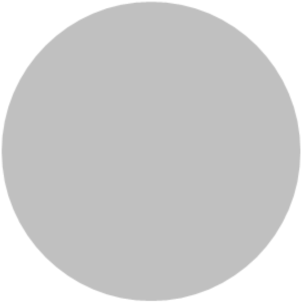


ARCHIMEDES
287 BC – 212 BC

HOW BIG IS THIS CIRCLE?

As the diameter increases, so does the circumference (obviously). *But what's the relationship between diameter and circumference?*

That's the question the great Archimedes set out to answer.



HOWEVER:

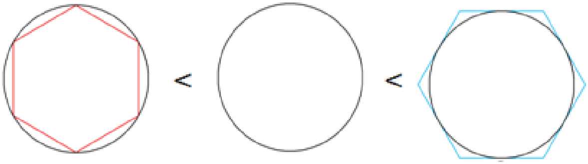
I forgot to tell you: you can only use a straight-edge ruler, and you don't know $C = 2\pi r$ because π hasn't been invented yet!

MEASURING THE CIRCLE

Estimation Using the "Sandwich Method"

If I inscribe a polygon in the circle,

If I inscribe the circle in a polygon,

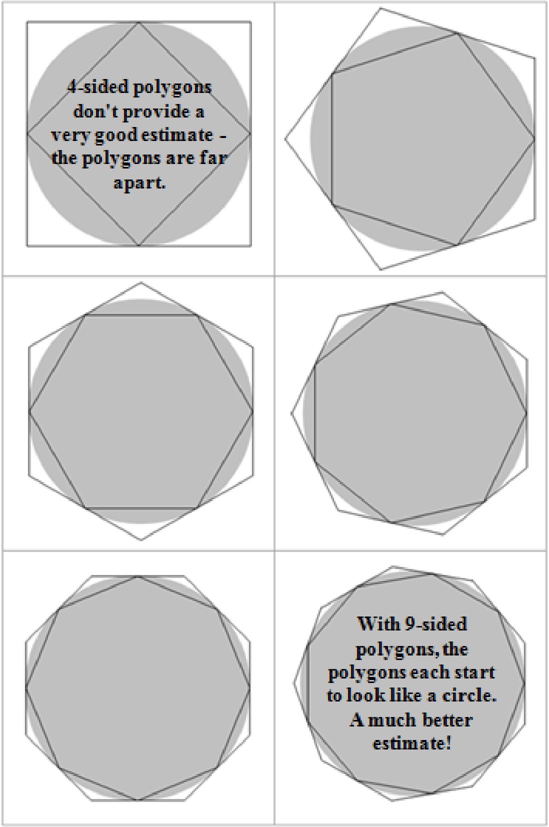


Then the circle is slightly *bigger* than the polygon's perimeter.

Then the circle is slightly *smaller* than the polygon's perimeter.

By measuring the perimeter of the inscribed and circumscribed polygons, I can average them to get an estimate.

BUT HOW GOOD IS THE ESTIMATE?


4-sided polygons don't provide a very good estimate - the polygons are far apart.
With 9-sided polygons, the polygons each start to look like a circle. A much better estimate!

The Inner Polygon

Finding the Perimeter

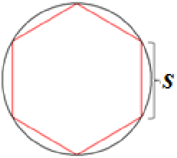
OUR EXAMPLE

To “Get Something on the Table”, let’s use a regular hexagon as our example.

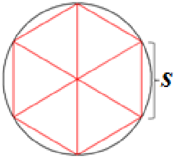
THE START OF A PLAN

Measuring the Perimeter of the Inner Polygon

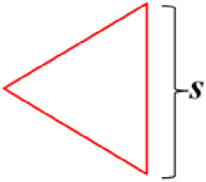
In finding the perimeter of the regular hexagon, I need to find the length of just one side, *s*



I can create six equal triangles.



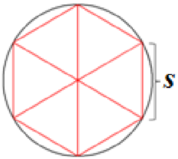
Because this one triangle is similiar to all six, I can concentrate on just this one!



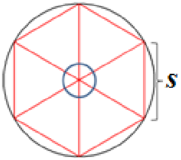
FINDING AN ANGLE

Measuring the Perimeter of the Inner Polygon

The regular hexagon can be split into six similar triangles.

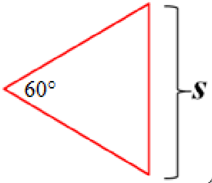


The center angles in total are equal to the measure of a circle, 360°.



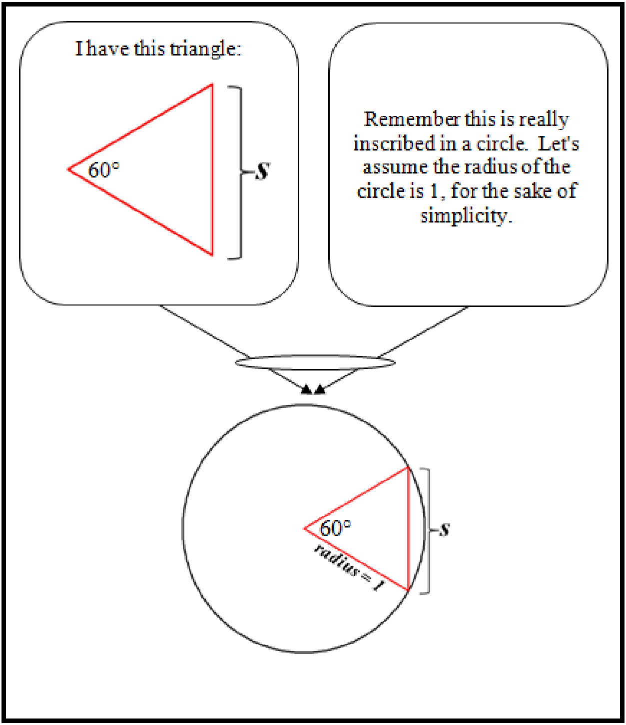
Each center angle is

$$\frac{360^\circ}{6} = 60^\circ$$



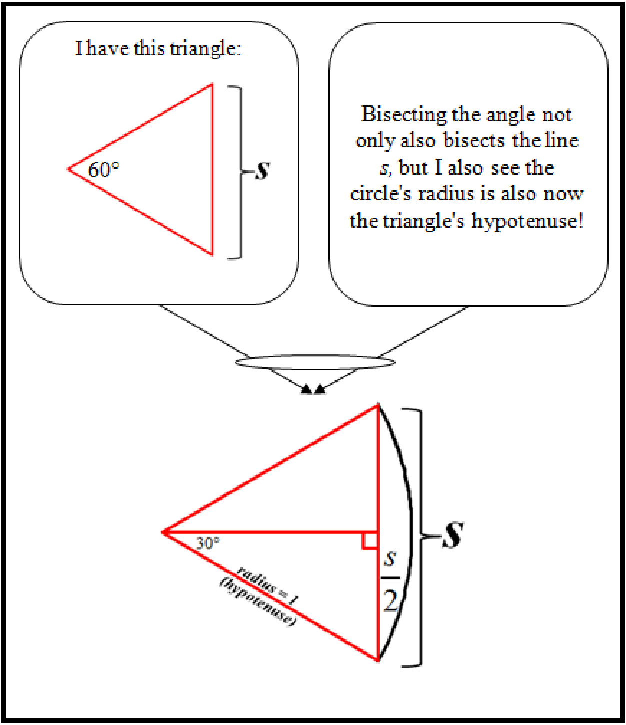
MORE INFORMATION

Measuring the Perimeter of the Inner Polygon



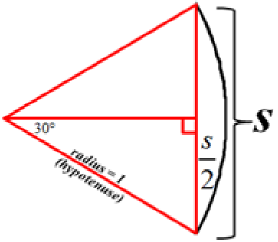
THE TRIANGLE

Measuring the Perimeter of the Inner Polygon



THE PERIMETER!

Measuring the Perimeter of the Inner Polygon



In Trigonometry:

SOHCAHTOA

$$Sin(\theta) = \frac{opposite}{hypotenuse}$$

$$\sin(30°) = \frac{s/2}{1} = \frac{s}{2}$$

$$2\sin(30°) = s$$

The hexagon has six sides.

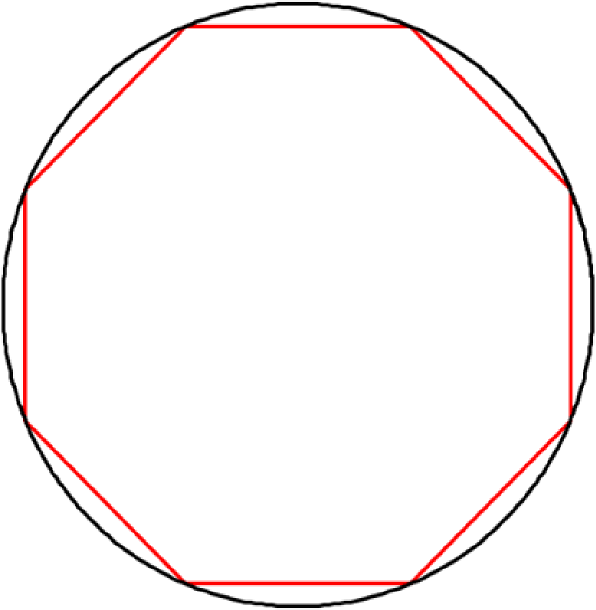
PERIMETER of the *inner* polygon

$$P_{inner} = 6(2\sin(30°))$$

$$= 12\sin(30°)$$

YOUR PROBLEM

Find the Perimeter of this Inscribed Regular Octagon

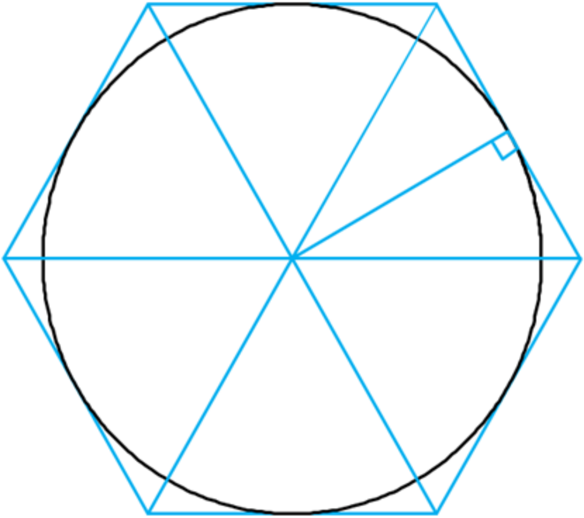


The Outer Polygon

Finding the Perimeter

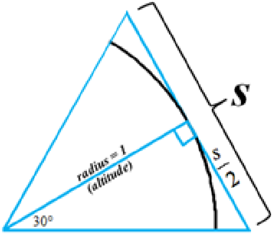
THE OUTER POLYGON

Label the Diagram, Similar to the Inner Polygon Process



THE OUTER POLYGON

We Have the Perimeter!



In Trigonometry:

SOHCAHTOA

$$Tan(\theta) = \frac{opposite}{adjacent}$$

$$\tan(30°) = \frac{s/2}{1} = \frac{s}{2}$$

$$2\tan(30°) = s$$

The hexagon has six sides.

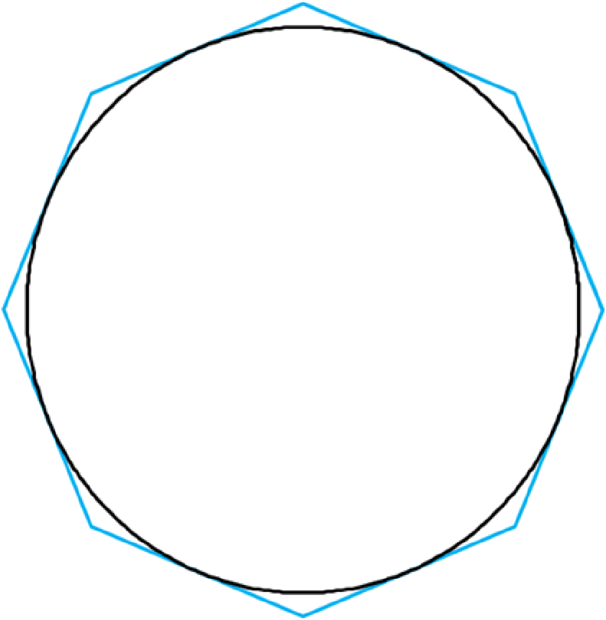
PERIMETER of the *outer* polygon

$$P_{outer} = 6(2\tan(30°))$$

$$= 12\tan(30°)$$

YOUR PROBLEM

Find the Perimeter of this Circumscribed Regular Octagon



π

An Estimation

ESTIMATING π

Averaging Our Earlier Results

PERIMETER of the inner polygon.

$$P_{inner} = 12\sin(30°)$$

PERIMETER of the outer polygon.

$$P_{outer} = 12\tan(30°)$$

$$P_{average} = \frac{12\sin(30°) + 12\tan(30°)}{2}$$

$$= \frac{12\left[\sin(30°) + \tan(30°)\right]}{2}$$

$$= 6\left[\sin(30°) + \tan(30°)\right]$$

$$= 6.464$$

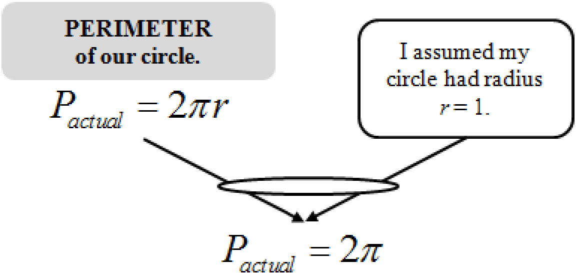
This is the "Average" Perimeter, but what does this number mean?

ESTIMATING π

The average perimeter of the polygons – my estimate of π – is 6.464? *What kind of estimate is this?* Where have I gone wrong? I know the formula for the circumference of a circle is:

$$\mathbf{C = 2\pi} \boldsymbol{r}$$

Since I'm using "perimeter" in the above calculations, let's use it here, too. And if my assumption was $r = 1$, I have:



PUTTING THIS ALL TOGETHER

Estimating the Value of π

AVERAGE PERIMETER
of the two polygons

$$P_{average} = 6\left[\sin(30°) + \tan(30°)\right]$$

ACTUAL DEFINITION
of a circle's circumference

$$P_{actual} = 2\pi$$

$$6\left[\sin(30°) + \tan(30°)\right] \approx 2\pi$$

$$3\left[\sin(30°) + \tan(30°)\right] \approx \pi$$

$$3.232 \approx \pi$$

OUR ESTIMATE!

THE GEOMETRIC MIND

PROBLEMS

The following three problems each have a CHECK (to make sure you've done the problem right).

Once you've confirmed you've done the problem right, there's a KEY. The key is necessary to unlock the next installment.

Key1 **Key2** **Key3**

PROBLEM 1

When 18-sided polygons are used, we get an estimate of π. When 36-sided polygons are used, we get a better estimate of π. *How much better?*

# of Sides for Estimate						
18-Sides		.			9	
36-Sides		.		4		
		.				2
					Key1	Check

PROBLEM 2

If we're using 30-sided polygons to find an estimate of π, what is the sin(relevant angle)?

0	**.**			**4**	
		Key2		Check	

PROBLEM 3

What is the least sided-polygon needed to estimate

$\pi \approx 3.14159$?

		9
Key3		Check

CONCEPT CARD